## §2a W-algebra

$\mathfrak{g}$ : cpx simple Lie algebra

$V_k(\mathfrak{g})$ : affine vertex algebra at level $k \in \mathbb{C}$

vaccum repr. of $\hat{\mathfrak{g}}$ at level $k$, i.e., $\mathrm{Ind}_{\mathfrak{g}[t]\oplus\mathbb{C}K}^{\hat{\mathfrak{g}}} \mathbb{C}_k$

( Heisenberg vertex algebra $\quad \mathrm{Fock} = \mathbb{C}[b_{-1}, b_{-2}, \cdots] \ni 1 = |0\rangle$

$b(z) = \sum b_n z^{-n-1} = Y(b_{-1}|0\rangle, z)$

more generally $\quad Y(v,z) \quad \forall v \in \mathrm{Fock}$

**Def.** $W_k(\mathfrak{g})$ = quantum Hamiltonian reduction of $V_k(\mathfrak{g})$ [Feigin–Frenkel]

$= H^0(V_k(\mathfrak{g}) \otimes Cl \ , \ d)$

fermionic vertex algebra $\quad$ differential

e.g. $\mathfrak{g} = \mathfrak{sl}_2 \qquad W_k(\mathfrak{g}) = Vir_{c(k)} \quad c(k) = \text{central charge} = 1 - 6\left(\frac{1}{k+2} - 2 + k+2\right)$

$\left( \begin{array}{l} 1 = \mathrm{rk}(\mathfrak{sl}_2) \\ 2 = h^\vee \text{ for } \mathfrak{sl}_2 \end{array} \right) = 1 - 6\left(\sqrt{k+2} - \frac{1}{\sqrt{k+2}}\right)^2$

$W_k(\mathfrak{g})$ has "generating" fields $W^{(i)}(z) \quad (i = 1, \cdots, \ell)$

corresponding to generators of invariant polynomials $S(\mathfrak{g})^{\mathfrak{g}} \cong S(\mathfrak{h})^W$

$\mathfrak{g} = \mathfrak{sl}_2 \longrightarrow W^{(1)}(z) = T(z) \longleftrightarrow$ Casimir

Rem. • $\mathfrak{g} = \mathfrak{sl}_3 \longrightarrow W_k(\mathfrak{g})$ is not a Lie algebra
• $\mathfrak{g}$ : bigger $\Longrightarrow$ OPE among $W^{(i)}(z)$ are complicated to be written down.
$\Longrightarrow$ impossible to define W-alg representation by generator/rel.

○ geometric intuition to $W_k(\mathfrak{g})$:

$k \to \infty$ $W_k(\mathfrak{g})$ : commutative $\cong \mathbb{C}[Op_G(D)]$

$Op_G(D)$ = moduli space of G-opers on the formal disk $D$

$$= \{ \nabla = d + \underbrace{p_-}_{\text{regular nilpotent} \in \mathfrak{n}_-} + \underbrace{A(t)}_{\mathfrak{b}_+} \overset{\text{e.g.}}{=} d + \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \} / N_+[[t]]$$

$$\cong \{ \nabla = d + p_- + A(t) \mid A(t) \in \underbrace{Z(p_-)}_{\text{centralizer}}[[t]] \} \leftarrow \infty\text{-dim'l vector space}$$

Kostant slice $\quad S(Z(p_-)) \cong Z(U(\mathfrak{g})) \underset{HC}{\cong} S(\mathfrak{h})^W \quad \therefore W_{k=\infty}(\mathfrak{g}) \sim S(\mathfrak{h}[[t]])^W$

$$\underbrace{Conn_G(C)}_{\text{moduli sp. of G-connection}} \supset \overset{\text{lagrangian}}{Op_G(C)} \quad \Big( \overset{\rightsquigarrow}{\text{degeneration}} \quad Higgs_G(C) \supset 0\text{-}Op_G(C) \ \text{lagrangian}$$

$$Higgs_G(C) \to Hitch_G(C) \xleftarrow{\cong} 0\text{-}Op_G(C)$$

$\therefore W_k(\mathfrak{g})$ = quantization of $Conn_G(D)$

Fact. [FF] ① $k$: generic $\quad W_k(\mathfrak{g}) \subset \mathrm{Heis}_{k+h^\vee}(\mathfrak{h})$ $\qquad \mathfrak{h}$: Cartan

$$[h^i_m, h^j_n] = m\,\delta_{m+n,0}\,(\alpha_i, \alpha_j) \times (k+h^\vee)$$

e.g. $\mathrm{Vir} \subset \mathrm{Heis}(\mathfrak{h}_{sl_2})$ $\quad T(z) = \frac{1}{k+2}\left[\frac{1}{4} : h(z)^2 : + \frac{1}{2}(1-(k+2))\,\partial_z h(z)\right]$

$\left( \begin{array}{l} \text{std} \\ \text{Heis.} \end{array} \; b(z) = \frac{h(z)}{\sqrt{2(k+2)}} \quad T = \frac{1}{2} : b(z)^2 : - \frac{1}{\sqrt{2}}\left(\sqrt{k+2} - \frac{1}{\sqrt{k+2}}\right)\partial_z b(z) \right)$

② $k$: generic $\qquad W_k(\mathfrak{g}) = \bigcap_{i \in I} \mathrm{Vir}_i \otimes \mathrm{Heis}(\alpha_i^\perp)$

One can consider this as a definition of the W-algebra

---

○ integral form

$A := \mathbb{C}[\varepsilon_1, \varepsilon_2] \qquad k + h^\vee = -\frac{\varepsilon_2}{\varepsilon_1}$

$\mathrm{Heis}_A(\mathfrak{h})$ = vertex $A$-subalg. generated by $\left\langle \tilde{P}^i_m = \varepsilon_1 h^i_m \;\middle|\; \begin{array}{l} i \in I \\ m \in \mathbb{Z} \end{array} \right\rangle$

$$[\tilde{P}^i_m, \tilde{P}^j_n] = -m\,\delta_{m+n,0}\,(\alpha_i, \alpha_j)\,\varepsilon_1\varepsilon_2$$

$\mathrm{Vir}_{i,A} := \langle \tilde{L}^i_m \rangle$

$$[\tilde{L}^i_m, \tilde{L}^i_n] = (m-n)\,\varepsilon_1\varepsilon_2\,\tilde{L}^i_{m+n} + \left\{(\varepsilon_1\varepsilon_2)^2 + 6\,\varepsilon_1\varepsilon_2(\varepsilon_1+\varepsilon_2)^2\right\}\delta_{m+n,0}\,\frac{m^3-m}{12}$$

$$W_A(\mathfrak{g}) := \bigcap_i \mathrm{Vir}_{i,A} \otimes \mathrm{Heis}_A(\alpha_i^\perp)$$

## §2b Uhlenbeck space

$G$ : complex reductive group, e.g. $SL_r$

$(\ ,\ )$ : invariant bilinear form on $\mathfrak{g}$

$\mathrm{Bun}_G^d$ : moduli space of $\underbrace{\wedge}_{\text{holomorphic}}$ $G$-bundles $\mathcal{E}$ over $\underset{\mathbb{C}^2 \cup \ell_\infty}{\overset{\shortparallel}{\mathbb{P}^2}}$ with framing $\varphi : \mathcal{E}|_{\ell_\infty} \xrightarrow{\sim} \ell_\infty \times G$

$d$ = instanton #

$\underset{\text{Bando}}{=}$ moduli space of $G_{cpt}$-instantons over $\underset{\mathbb{R}^4 \cup \infty}{\overset{\shortparallel}{S^4}}$ with framing $\varphi : E_\infty \cong G$

instanton # = characteristic class corresponding to $(\ ,\ )$

e.g. $G = SL_r$, $(\ ,\ )$ : std $\Rightarrow$ $c_2$ : second Chern class

<u>Fact</u> $\mathrm{Bun}_G^d$ : smooth, quasi proj variety ($G$ : simple, $(\theta, \theta) = 2$

$\dim = 2dh^\vee$ $\quad h^\vee$ : dual Coxeter #

We can define its partial cptification :

$\mathcal{U}_G^d$ : Uhlenbeck space $\quad$ affine variety $\quad$ [Braverman-Finkelberg-Gaitsgory]

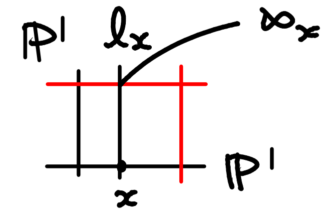
$$= \coprod_{d = d' + |\lambda|} \mathcal{U}_G^{d'} \times S_\lambda \mathbb{C}^2 \qquad \text{stratification}$$

For $G = SL_r$, we have a better space

$\widetilde{\mathcal{U}}_r^d$ : Gieseker space = moduli space of torsion free sheaves on $\mathbb{P}^2$ + framing

$$\pi : \widetilde{\mathcal{U}}_r^d \longrightarrow \mathcal{U}_{SL_r}^d$$

<u>Fact</u>
- resolution of singularities
- quiver variety for Jordan quiver




## Alternative description

Observe $\mathrm{Bun}_G^d = \{\text{framed } G\text{-b'dles } \mathcal{E} \text{ on } \mathbb{P}^1 \times \mathbb{P}^1\}$

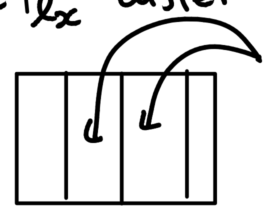
$\therefore\ \mathcal{E}|_{l_x}$ : $G$-b'dles on $l_x \cong \mathbb{P}^1$ trivialised at $\infty_x$

$\in \mathcal{G}_G$ affine Grassmannian

$\therefore\ \mathrm{Bun}_G^d = \mathrm{Map}_*^d(\mathbb{P}^1, \mathcal{G}_G)$ $\quad * = $ based $\quad \infty \mapsto \text{triv.} \in \mathcal{G}_G$

## Factorization

Take $a : \mathbb{C}^2 \to \mathbb{C}^1$ projection. We can define $\pi_a : \mathcal{U}_G^d \to S^d\mathbb{C}^1$, which measures how $\mathcal{E}|_{l_x}$ differ from a trivial $G$-b'dle



$\mathcal{E}|_{l_x}$ is trivial outside $\pi_a(\mathcal{E})$

$\times \ \times \ \times \ \leftarrow \pi_a(\mathcal{E})$

Let

$$(S^{d_1}\mathbb{C}^1 \times S^{d_2}\mathbb{C}^1)_0 \underset{\text{open}}{\subset} S^{d_1}\mathbb{C}^1 \times S^{d_2}\mathbb{C}^1 \quad \text{disjoint support}$$

$$\searrow S^d\mathbb{C}^1 \quad d = d_1 + d_2$$

$$\Longrightarrow \quad \mathcal{U}_G^d \times_{S^d\mathbb{C}^1} (S^{d_1}\mathbb{C}^1 \times S^{d_2}\mathbb{C}^1)_0 \cong (\mathcal{U}_G^{d_1} \times \mathcal{U}_G^{d_2}) \times_{S^{d_1}\mathbb{C}^1 \times S^{d_2}\mathbb{C}^1} (S^{d_1}\mathbb{C}^1 \times S^{d_2}\mathbb{C}^1)_0$$

$\therefore$ Only need to understand $\pi_a^{-1}(S_{(d)}\mathbb{C}^1)$

○ restriction to Levi

$\lambda : \mathbb{C}^* \longrightarrow G$ hom

$G^{\lambda(\mathbb{C}^*)} = \{ g \in G \mid \lambda(t) g \lambda(t)^{-1} = g \} = L$ : Levi subgrp

e.g. $\lambda(t) = \begin{bmatrix} t^{n_1} & & 0 \\ & t^{n_2} & \\ 0 & & \ddots \ t^{n_\ell} \end{bmatrix}$ $n_i$ : distinct $\Rightarrow L = \begin{bmatrix} * & & 0 \\ & * & \\ 0 & & \ddots \ * \end{bmatrix}$

$P$ : parabolic subgrp $= \{ g \in G \mid \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1}$ exists $\}$

$n_1 > n_2 > \cdot > n_\ell \Rightarrow P = \begin{bmatrix} \ulcorner & * \\ 0 & \lrcorner \end{bmatrix}$

$\therefore G \supset P \overset{\supseteq}{\twoheadrightarrow} L$

└ quotient given by $\lim_{t \to 0}$

Consider the corresponding subvarieties for $\mathcal{U}_G^d$

$G \curvearrowright \mathcal{U}_G^d$ by change of the framing

$\therefore \mathbb{C}^* \curvearrowright \mathcal{U}_G^d$ via $\lambda : \mathbb{C}^* \longrightarrow G$

$(\mathcal{U}_G^d)^{\lambda(\mathbb{C}^*)} = \mathcal{U}_L^d$ (at least topologically)

$\mathcal{U}_P^d := \{ \mathcal{E} \in \mathcal{U}_G^d \mid \lim_{t \to 0} \lambda(t) \mathcal{E}$ exists $\}$

$\mathcal{U}_G^d \hookleftarrow \mathcal{U}_P^d \underset{\longrightarrow}{\hookleftarrow} \mathcal{U}_L^d$

**Remark** $\mathcal{U}_P^d$ is **not** necessarily consisting of framed $P$-bdles

**ex.** $0 \to E_1 \to E_2 \to \mathcal{O} \to \mathbb{C}_x \to 0$ Koszul resol. of skyscraper sheaf at $x \in \mathbb{C}^2$

$\quad\quad\quad$ $E_1$: rk 1, $E_2$: rk 2

$$0 \to E_1 \to E_2 \to \mathcal{I}_x \to 0$$

$$\therefore \lim_{t \to 0} \lambda(t) E_2 = \mathrm{triv.}^{\oplus 2} + x \in \mathcal{U}^1_{\mathbb{C}^* \times \mathbb{C}^*} = S^1\mathbb{C}^2$$

$\therefore E_2 \in \mathcal{U}_P^1 \cap \mathrm{Bun}^1_{SL_2}$, but not a $P$-bdle

If $\lambda$: generic $\Rightarrow \begin{cases} L = T \\ P = B \end{cases}$

**Note** No nontrivial $T$-bdles with framing

$$\therefore \mathcal{U}_T^d = S^d\mathbb{C}^2$$